CSC D70: Compiler Optimization
Register Allocation

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The content of this lecture is adapted from the lectures of Todd Mowry and Phillip Gibbons
Register Allocation and Coalescing

- Introduction
- Abstraction and the Problem
- Algorithm
- Spilling
- Coalescing

Reading: ALSU 8.8.4
Motivation

• **Problem**
  – Allocation of variables (pseudo-registers) to hardware registers in a procedure

• **A very important optimization!**
  – Directly reduces running time
    • (memory access \(\rightarrow\) register access)
  – Useful for other optimizations
    • e.g. CSE assumes old values are kept in registers.
Goals

• Find an allocation for all pseudo-registers, if possible.

• If there are not enough registers in the machine, choose registers to spill to memory
Register Assignment Example

- Find an assignment (no spilling) with only 2 registers
  - A and D in one register, B and C in another one
- What assumptions?
  - After assignment, no use of A & (and only one of B and C used)
An Abstraction for Allocation & Assignment

• **Intuitively**
  – Two pseudo-registers *interfere* if at some point in the program they cannot both occupy the same register.

• **Interference graph**: an undirected graph, where
  – nodes = pseudo-registers
  – there is an edge between two nodes if their corresponding pseudo-registers interfere

• **What is not represented**
  – Extent of the interference between uses of different variables
  – Where in the program is the interference

   Interfere many times vs. once

   E.g., cold path vs. hot path
Register Allocation and Coloring

• A graph is **n-colorable** if:
  – every node in the graph can be colored with one of the n colors such that two adjacent nodes do not have the same color.

• Assigning n register (without spilling) = Coloring with n colors
  – assign a node to a register (color) such that no two adjacent nodes are assigned same registers (colors)

• Is spilling necessary? = Is the graph n-colorable?

• To determine if a graph is n-colorable is **NP-complete, for n>2**
  – Too expensive
  – Heuristics
Algorithm

Step 1. Build an interference graph
   a. refining notion of a node
   b. finding the edges

Step 2. Coloring
   – use heuristics to try to find an n-coloring
     • Success:
       – colorable and we have an assignment
     • Failure:
       – graph not colorable, or
       – graph is colorable, but it is too expensive to color
Step 1a. Nodes in an Interference Graph

A = ...
IF A goto L1

B = ...
= A
D =
= B + D

L1: C = ...
= A
D =
= D + C

A = 2

= A

Interference Graph

Should we add A-D edge?
No, since new def of A
Live Ranges and Merged Live Ranges

- Motivation: to create an interference graph that is easier to color
  - Eliminate interference in a variable’s “dead” zones.
  - Increase flexibility in allocation:
    - can allocate same variable to different registers
- A **live range** consists of a definition and all the points in a program in which that definition is live.
  - How to compute a live range?
- Two overlapping live ranges for the **same** variable must be merged

```
  a = ...      a = ...
    \------------
      \......=a
```
Example (Revisited)

Live Variables

Reaching Definitions

\[
\begin{align*}
A &= \ldots (A_1) \\
\text{IF } A \text{ goto L1} \\
L1: \quad C &= \ldots (C_1) \\
&= A \\
D &= \ldots (D_1) \\
\end{align*}
\]

\[
\begin{align*}
A &= 2 \quad (A_2) \\
B &= \ldots (B_1) \\
&\quad = A \\
D &= B \quad (D_2) \\
\end{align*}
\]

\[
\begin{align*}
\{A\} &\quad \{A_1\} \\
\{A,B\} &\quad \{A_1, B_1\} \\
\{B\} &\quad \{A_1, B_1\} \\
\{D\} &\quad \{A_1, B_1, D_2\} \\
\end{align*}
\]

\[
\begin{align*}
\{D\} &\quad \{A_1, B_1, C_1, D_1, D_2\} \\
\{A,D\} &\quad \{A_2, B_1, C_1, D_1, D_2\} \\
\{D\} &\quad \{A_2, B_1, C_1, D_1, D_2\} \\
\end{align*}
\]

\[
\begin{align*}
\{A,D\} &\quad \{A_2, B_1, C_1, D_1, D_2\} \\
\{D\} &\quad \{A_2, B_1, C_1, D_1, D_2\} \\
\end{align*}
\]

\[
\begin{align*}
= A \\
\text{ret } D \\
\end{align*}
\]
Merging Live Ranges

• **Merging definitions into equivalence classes**
  – Start by putting each definition in a different equivalence class
  – Then, for each point in a program:
    • if (i) variable is live, and (ii) there are multiple reaching definitions for the variable, then:
      – merge the equivalence classes of all such definitions into one equivalence class
    • **Sounds familiar?**

• **From now on, refer to merged live ranges simply as live ranges**
  – merged live ranges are also known as “webs”
SSA Revisited: What Happens to \( \Phi \) Functions

• Now we see why it is unnecessary to “implement” a \( \Phi \) function
  – \( \Phi \) functions and SSA variable renaming simply turn into merged live ranges

• When you encounter: \( x_4 = \Phi(x_1, x_2, x_3) \)
  – merge \( x_1, x_2, x_3, \) and \( x_4 \) into the same live range
  – delete the \( \Phi \) function

• Now you have effectively converted back out of SSA form
Step 1b. Edges of Interference Graph

• **Intuitively:**
  – Two live ranges (necessarily of different variables) may **interfere** if they overlap at some point in the program.
  – Algorithm:
    • At each point in the program:
      – enter an **edge for every pair of live ranges at that point.**

• **An optimized definition & algorithm for edges:**
  – Algorithm:
    • check for interference only at the start of each live range
    – Faster
    – Better quality
Live Range Example 2

Because ranges overlap: Won’t assign A and B to same register (even though would have been ok: path sensitive vs. path insensitive analysis)
Step 2. Coloring

• **Reminder:** *coloring for n > 2 is NP-complete*

• **Observations:**
  – a node with degree < n ⇒
    • can always color it successfully, given its neighbors’ colors
  – a node with degree = n ⇒
    • can only color if at least two neighbors share same color
  – a node with degree > n ⇒
    • maybe, not always
Coloring Algorithm

- **Algorithm:**
  - Iterate until stuck or done
    - Pick any node with degree < n
    - Remove the node and its edges from the graph
  - If done (no nodes left)
    - reverse process and add colors
- **Example (n = 3):**

- **Note:** degree of a node may drop in iteration
- Avoids making arbitrary decisions that make coloring fail
More details

• **Apply coloring heuristic**
  
  Build interference graph
  
  Iterate until there are no nodes left
  
  If there exists a node \( v \) with less than \( n \) neighbor
  
  push \( v \) on register allocation stack
  
  else
  
  return (coloring heuristics fail)
  
  remove \( v \) and its edges from graph
  
• **Assign registers**
  
  While stack is not empty
  
  Pop \( v \) from stack
  
  Reinsert \( v \) and its edges into the graph
  
  Assign \( v \) a color that differs from all its neighbors
What Does Coloring Accomplish?

• **Done:**
  – colorable, also obtained an assignment

• **Stuck:**
  – colorable or not?
Extending Coloring: Design Principles

• **A pseudo-register is**
  - Colored successfully: allocated a hardware register
  - Not colored: left in memory

• **Objective function**
  - Cost of an uncolored node:
    • proportional to number of uses/definitions (dynamically)
    • estimate by its loop nesting
  - Objective: minimize sum of cost of uncolored nodes

• **Heuristics**
  - Benefit of spilling a pseudo-register:
    • increases colorability of pseudo-registers it interferes with
    • can approximate by its degree in interference graph
  - Greedy heuristic
    • spill the pseudo-register with lowest cost-to-benefit ratio, whenever spilling is necessary
Spilling to Memory

- **CISC architectures**
  - can operate on data in memory directly
  - memory operations are slower than register operations

- **RISC architectures**
  - machine instructions can only apply to registers
    - Use
      - must first load data from memory to a register before use
    - Definition
      - must first compute RHS in a register
      - store to memory afterwards
  - Even if spilled to memory, needs a register at time of use/definition
Chaitin: Coloring and Spilling

• **Identify spilling**
  Build interference graph
  Iterate until there are no nodes left
  - If there exists a node v with less than n neighbor
    place v on stack to register allocate
  else
    v = node with highest degree-to-cost ratio
    mark v as spilled
    remove v and its edges from graph

• **Spilling may require use of registers; change interference graph**
  While there is spilling
  rebuild interference graph and perform step above

• **Assign registers**
  While stack is not empty
  Remove v from stack
  Reinsert v and its edges into the graph
  Assign v a color that differs from all its neighbors
Spilling

• **What should we spill?**
  - Something that will eliminate a lot of interference edges
  - Something that is used infrequently
  - Maybe something that is live across a lot of calls?

• **One Heuristic:**
  - spill cheapest live range (aka “web”)
  - Cost = [(#defs & uses) * 10^{loop-nest-depth}] / degree
Quality of Chaitin’s Algorithm

- Giving up too quickly

- N=2

- An optimization: “Prioritize the coloring”
  - Still eliminate a node and its edges from graph
  - Do not commit to “spilling” just yet
  - Try to color again in assignment phase.
Splitting Live Ranges

• **Recall**: Split pseudo-registers into live ranges to create an interference graph that is easier to color
  
  – Eliminate interference in a variable’s “dead” zones.
  
  – Increase flexibility in allocation:
    • can allocate same variable to different registers
Insight

• Split a live range into smaller regions (by paying a small cost) to create an interference graph that is easier to color
  – Eliminate interference in a variable’s “nearly dead” zones.
    • Cost: Memory loads and stores
      – Load and store at boundaries of regions with no activity
    • # active live ranges at a program point can be > # registers

– Can allocate same variable to different registers
  • Cost: Register operations
    – a register copy between regions of different assignments
  • # active live ranges cannot be > # registers
Examples

Example 1:

FOR $i = 0$ TO 10
    FOR $j = 0$ TO 10000
        $A = A + \ldots$
        *(does not use $B$)*
    FOR $j = 0$ TO 10000
        $B = B + \ldots$
        *(does not use $A$)*

Example 2:

```
a =

b =
    = a + b

b =
    = b + c

c =
    = a + c
```

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Example 1

FOR $i = 0$ TO $10$
  FOR $j = 0$ TO $10000$
    $A = A + \ldots$
    *(does not use $B$)*
  
  FOR $j = 0$ TO $10000$
    $B = B + \ldots$
    *(does not use $A$)*

Diagram:

- $n=2$
- Nodes: A, B, i, j
- Connections: A to B, A to i, A to j, B to i, B to j, i to j
Example 2
Live Range Splitting

• When do we apply live range splitting?
• Which live range to split?
• Where should the live range be split?
• How to apply live-range splitting with coloring?
  – Advantage of coloring:
    • defers arbitrary assignment decisions until later
  – When coloring fails to proceed, may not need to split live range
    • degree of a node >= n does not mean that the graph definitely is not colorable
  – Interference graph does not capture positions of a live range
One Algorithm

• **Observation**: spilling is absolutely necessary if
  – number of live ranges active at a program point > n

• **Apply live-range splitting before coloring**
  – Identify a point where number of live ranges > n
  – For each live range active around that point:
    • find the outermost “block construct” that does not access the variable
  – Choose a live range with the largest inactive region
  – Split the inactive region from the live range
Summary

• Problems:
  – Given n registers in a machine, is spilling avoided?
  – Find an assignment for all pseudo-registers, whenever possible.

• Solution:
  – Abstraction: an interference graph
    • nodes: live ranges
    • edges: presence of live range at time of definition
  – Register Allocation and Assignment problems
    • equivalent to \textit{n-colorability} of interference graph
      ➔ NP-complete
  – Heuristics to find an assignment for n colors
    • successful: colorable, and finds assignment
    • not successful: colorability unknown & no assignment
CSC D70:
Compiler Optimization
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Let’s Focus on Copy Instructions

- Optimizations that help optimize away copy instructions:
  - Copy Propagation
  - Dead Code Elimination

- Can all copy instructions be eliminated using this pair of optimizations?
Example Where Copy Propagation Fails

- Use of copy target has multiple (conflicting) reaching definitions

\[ X = A + B; \]
\[ Y = C; \]
\[ Y = X; \]
\[ Z = Y + 4; \]
Another Example Where the Copy Instruction Remains

- Copy target \( Y \) still live even after some successful copy propagations

- **Bottom line:**
  - copy instructions may still exist when we perform register allocation
Copy Instructions and Register Allocation

• What clever thing might the register allocator do for copy instructions?

• If we can assign both the source and target of the copy to the same register:
  – then we don’t need to perform the copy instruction at all!
  – the copy instruction can be removed from the code
    • even though the optimizer was unable to do this earlier

• One way to do this:
  – treat the copy source and target as the same node in the interference graph
    • then the coloring algorithm will naturally assign them to the same register
  – this is called “coalescing”
Simple Example: Without Coalescing

X = ...;
A = 5;
Y = X;
B = A + 2;
Z = Y + B;
return Z;

- Without coalescing, X and Y can end up in different registers
  - cannot eliminate the copy instruction
Example Revisited: With Coalescing

- With coalescing, \( X \) and \( Y \) are now guaranteed to end up in the same register
  - the copy instruction can now be eliminated

- Great! So should we go ahead and do this for every copy instruction?

```
X = ...;
A = 5;
Y = X;  \( \text{\textcolor{red}{\underline{\text{X}}} \text{\textcolor{red}{\underline{\text{Y}}}}} \)
B = A + 2;
Z = Y + B;
return Z;
```

Valid coloring with 3 registers
Should We Coalesce \( X \) and \( Y \) In This Case?

- It is legal to coalesce \( X \) and \( Y \) for a \( Y = X \) copy instruction iff:
  - initial definition of \( Y \)'s live range is this copy instruction, AND
  - the live ranges of \( X \) and \( Y \) do not interfere otherwise

- But just because it is legal doesn’t mean that it is a good idea...

No! That would result in incorrect behavior if this branch is taken.

\[
\begin{align*}
X &= A + B; \\
Y &= X; \\
Z &= Y + X; \\
X &= 2;
\end{align*}
\]
Why Coalescing May Be Undesirable

What is the likely impact of coalescing $X$ and $Y$ on:
- live range size(s)?
  - recall our discussion of live range splitting
  - colorability of the interference graph?

Fundamentally, coalescing adds further constraints to the coloring problem
- doesn’t make coloring easier; may make it more difficult

If we coalesce in this case, we may:
- save a copy instruction, BUT
- cause significant spilling overhead if we can no longer color the graph

```plaintext
X = A + B;
...
// 100 instructions
Y = X;
...
// 100 instructions
Z = Y + 4;
```
When to Coalesce

• Goal when coalescing is legal:
  – coalesce *unless* it would make a colorable graph non-colorable

• The bad news:
  – predicting colorability is tricky!
    • it depends on the shape of the graph
    • graph coloring is NP-hard

• Example: assuming 2 registers, should we coalesce X and Y?

![Graphs showing 2-colorability and non-2-colorability](image-url)
Representing Coalescing Candidates in the Interference Graph

- To decide whether to coalesce, we augment the interference graph.
- Coalescing candidates are represented by a new type of interference graph edge:
  - dotted lines: coalescing candidates
    - *try* to assign vertices the same color
      - (unless that is problematic, in which case they can be given different colors)
  - solid lines: interference
    - vertices *must* be assigned different colors

```plaintext
X = ...;
A = 5;
Y = X;
B = A + 2;
Z = Y + B;
return Z;
```
How Do We Know When Coalescing Will Not Cause Spilling?

• **Key insight:**
  – Recall from the coloring algorithm:
    • we can always successfully N-color a node if its degree is < N

• To ensure that **coalescing does not cause spilling:**
  – check that the degree < N invariant is still locally preserved after coalescing
    • if so, then coalescing won’t cause the graph to become non-colorable
  – no need to inspect the entire interference graph, or do trial-and-error

• **Note:**
  – We do NOT need to determine whether the full graph is colorable or not
  – Just need to check that coalescing does not cause a colorable graph to become non-colorable
Simple and Safe Coalescing Algorithm

- We can safely coalesce nodes $X$ and $Y$ if $|X| + |Y| < N$
  - Note: $|X|$ = degree of node $X$ counting interference (not coalescing) edges
- Example:
  - if $N \geq 4$, it would always be safe to coalesce these two nodes
    - this cannot cause new spilling that would not have occurred with the original graph
  - if $N < 4$, it is unclear

$|X| + |Y| = (1 + 2) = 3$

Degree of coalesced node can be no larger than 3

How can we (safely) be more aggressive than this?
What About This Example?

- Assume $N = 3$
- Is it safe to coalesce $X$ and $Y$?

  - Notice: $X$ and $Y$ share a common (interference) neighbor: node $A$
    - hence the degree of the coalesced $X/Y$ node is actually 2 (not 3)
    - therefore coalescing $X$ and $Y$ is guaranteed to be safe when $N = 3$
  - How can we adjust the algorithm to capture this?

\[
(|X| + |Y|) = (1 + 2) = 3
\]

(Not less than $N$)
Another Helpful Insight

• **Colors are not assigned until nodes are popped off the stack**
  – nodes with degree < N are pushed on the stack first
  – when a node is popped off the stack, we know that it can be colored
    • because the number of potentially conflicting neighbors must be < N

• **Spilling only occurs if there is no node with degree < N to push on the stack**

• **Example**: (N=2)
Another Helpful Insight

|X| = 5
|Y| = 5

2-colorable after coalescing X and Y?
Building on This Insight

• When would coalescing cause the stack pushing (aka “simplification”) to get stuck?
  1. coalesced node must have a degree \( \geq N \)
     • otherwise, it can be pushed on the stack, and we are not stuck
  2. AND it must have at least \( N \) neighbors that each have a degree \( \geq N \)
     • otherwise, all neighbors with degree \( < N \) can be pushed before this node
       – reducing this node’s degree below \( N \) (and therefore we aren’t stuck)

• To coalesce more aggressively (and safely), let’s exploit this second requirement
  – which involves looking at the degree of a coalescing candidate’s neighbors
    • not just the degree of the coalescing candidates themselves
Briggs’s Algorithm

• Nodes $X$ and $Y$ can be coalesced if:
  – (number of neighbors of $X/Y$ with degree $\geq N) < N$

• Works because:
  – all other neighbors can be pushed on the stack before this node,
  – and then its degree is $< N$, so then it can be pushed
  – **Example:** $(N = 2)$
Briggs’s Algorithm

- Nodes $X$ and $Y$ can be coalesced if:
  - (number of neighbors of $X/Y$ with degree $\geq N) < N$

- More extreme example: ($N = 2$)
George’s Algorithm

Motivation:
• imagine that $X$ has a very high degree, but $Y$ has a much smaller degree
  – (perhaps because $X$ has a large live range)

• With Briggs’s algorithm, we would inspect all neighbors both $X$ and $Y$
  – but $X$ has a lot of neighbors!
• Can we get away with just inspecting the neighbors of $Y$?
  – showing that coalescing makes coloring no worse than it was given $X$?
George’s Algorithm

- Coalescing $X$ and $Y$ does no harm if:
  - foreach neighbor $T$ of $Y$, either:
    1. degree of $T$ is $<N$, or $\leftarrow$ similar to Briggs: $T$ will be pushed before $X/Y$
    2. $T$ interferes with $X$ $\leftarrow$ hence no change compared with coloring $X$

- Example: ($N=2$)
Summary

- **Coalescing** can enable register allocation to **eliminate copy instructions**
  - if both source and target of copy can be allocated to the same register
- However, coalescing must be applied with care to **avoid causing register spilling**
- Augment the interference graph:
  - dotted lines for coalescing candidate edges
  - try to allocate to same register, unless this may cause spilling
- **Coalescing Algorithms**:
  - simply based upon degree of coalescing candidate nodes ($X$ and $Y$)
  - Briggs’s algorithm
    - look at degree of neighboring nodes of $X$ and $Y$
  - George’s algorithm
    - asymmetrical: look at neighbors of $Y$ (degree and interference with $X$)